Classical spin contribution to the Tunnel Effect

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Abstract

Since the spin of real particles is of order of $\hbar$, it is difficult to distinguish in a quantum mechanical experiment involving spinning particles what part of the outcome is related to the spin contribution and what part is a pure quantum mechanical effect. We analyze in detail a classical model of a nonrelativistic spinning particle under the action of a potential barrier and compute numerically the crossing for different potentials. In this way it is shown that because of the spin structure there is a nonvanishing contribution to crossing for energies above a certain minimum value, even below the top of the potential barrier. Results are compared with the quantum tunnel effect.

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I. INTRODUCTION

The aim of this work is to describe in classical terms the classical spin contribution to the so called tunnel effect by assuming a particular spin structure of a classical spinning particle.

In section II we make some general considerations about the classical limit of quantum mechanics, and also about the definition of elementary particle in quantum mechanics and the different treatment deserved to spin in classical as well as in quantum mechanics. These considerations lead us to propose a general definition of elementary particle at the classical level that resembles the one used in quantum mechanics.

A Lagrangian approach of a nonrelativistic spinning particle model to be used later, is described in detail in Sec. III where the main features are that the particle spin arises because the separation between its center of mass and center of charge. The center of mass satisfies Newton's equations where the external force is defined on the point charge position while this center of charge moves around the center of mass with an isotropic harmonic motion giving rise to the magnetic properties of the particle. The canonical approach of this model is also presented in Sec. IV.

We shall consider the solutions of a one-dimensional model of this classical spinning particle interacting with a potential barrier in Section V. Dynamical equations are nonlinear and a general solution is dificult to obtain in closed form even when the external potential is a linear function of the charge position. We solve the problem by numerical integration showing the penetration effect for different initial conditions and different slopes of the potential barrier. The crossing is achieved because of the separation between the center of mass and center of charge in such a way that the average repulsive force acting on the spinning particle is smaller than in the point particle case. Results are compared with the quantum tunnel effect for a point particle under the same one-dimensional potential.

II. CLASSICAL LIMIT OF QUANTUM MECHANICS

One usually reads in textbooks on quantum mechanics that *Classical Mechanics* (CM) *is the limit $\hbar \to 0$ of Quantum Mechanics* (QM) [1]. Of course there are exceptions. For instance in Feynman's Lectures on Physics [2] we read: '*In the classical limit, the quantum mechanics will agree with Newtonian Mechanics*'. Both expressions are equivalent if by CM we understand the classical mechanics of spinless (or point) particles, i.e., Newtonian Mechanics. We think this is the meaning the mentioned authors try to express. But the first statement as it stands, might lead to wrong interpretations if considered literally, as some of the quoted references might suggest. In order to clarify this idea of the classical limit of QM, let us consider the following simplified diagram of Fig. 1.

The set A represents the whole body of knowledge of CM which includes two subsets, the subset $(s = 0)$ or domain of spinless particles or Newtonian Mechanics and the subset $(s \neq 0)$ of spinning particles. If we restrict CM to satisfy the additional requirement of the Uncertainty Principle we enter into the more restricted body of knowledge of QM, represented by the smaller set B in which we also have the two subdomains of spinless and spinning particles. If in this more constrained domain we perform now the additional restriction of taking the limit $\hbar \to 0$ it is out of logic that after these two restrictions we shall reach the wider and less restrictive domain of the whole CM.

In QM we have that the measurement of any two observables C and D is not in general compatible, and the uncertainty in their simultaneous measurement is related to its commutator $[C, D] \simeq \hbar$ wich is of order of Planck's constant $\hbar$. But we also have in QM that the particle states satisfy eigenvalue equations for the spin of the form $S^2|\psi> = s(s+1)\hbar^2|\psi>$, where the right hand side is also a function of $\hbar$. It turns out that when performing the limit $\hbar \to 0$ we get $[C, D] = 0$ and also $S^2|\psi> = 0$, i.e., the physics of compatible observables of spinless systems.

With this analysis we see that the limit $\hbar \to 0$ of QM is in fact Newtonian Mechanics and not the whole domain of CM. Perhaps one of the reasons for the identification of CM

with Newtonian Mechanics in the mentioned references lies in the fact that we are used to work in QM with spinning systems since the early days of the quantum theory while the CM of spinning particles is still waiting for a complete development and improvement at least equivalent to the one we have achieved in the quantum domain. Even we can remember here that for many years, spin has been considered by physicists a strict quantum mechanical and relativistic property of the electron, as was pointed out by Levy-Leblond's detailed account [3] where the relevant references on this matter can be found. It is not strange that the recent history of physics had forgotten and considered unexistant the $(s \neq 0)$ region of CM. The spin is neither relativistic nor a quantum mechanical property of the electron. The only quantum mechanical aspect of the electron spin is that it is quantized.

In QM spin is usually defined as the translationally invariant part of the total angular momentum operator, or in an equivalent way, as the total angular momentum measured in the center of mass frame of the particle, when this frame is available. In the description of photons we have no center of mass frame but the first definition still works. But this definition is related to the representation of the generators of the rotation group on the Hilbert space of states. Even more, the QM spin dynamics is not postulated, it is contained in its commutation relations with the total Hamiltonian.

However, in CM spin is usually defined by assuming new degrees of freedom, the spin degrees of freedom, and even a plausible dynamical equation for the spin evolution. Let us mention the case of the Bargmann, Michel and Telegdi equation [4].

Following with the above scheme of Fig. 1 we see that physics first entered into the QM $(s \neq 0)$ region by hand. Remember the Pauli method of introducing σ-spin matrices into Schroedinger's equation, or for example when Dirac performs the linearization of Klein-Gordon operator to obtain the quantum mechanical relativistic description of the electron. But it is after Wigner's work [5] on the inhomogeneous Lorentz group representations, that by finding the different representations of the Poincaré group, we can obtain the description of particles of any spin. But Wigner's work not only develops the QM of spinning systems but it also provides a very precise mathematical definition of the concept of elementary

particle at the quantum level. An elementary particle is defined as a quantum mechanical system whose Hilbert space of pure states is the representation space of a projective unitary irreducible representation of the Poincaré group.

The very important expression with physical consequences, of the above mathematical definition lies in the term *irreducible*. Mathematically it means that the Hilbert space is an invariant vector space under the group action and that it has no other invariant subspaces. But it also means that there are no other states for a single particle that those that can be obtained by just taking any arbitrary vector state, form all its posible images in the different inertial frames and finally produce all linear combinations of these vectors.

We see that starting from a single state and by a simple change of inertial observer, we obtain the state of the particle described in this new frame. Then let us take the orthogonal part of this vector to the previous one and normalize it. Repeat this operation with another kinematical transformation acting on the same first state followed by the corresponding orthonormalization procedure, as many times as neccesary to finally obtain a complete orthonormal basis of the whole Hilbert space of states. All states in this basis are characterized by the physical parameters that define the first state and a countable collection of group transformations of the kinematical group G. And this can be done starting from any arbitrary state.

This idea allows us to define a concept of physical equivalence among states of any arbitrary quantum mechanical system in the following way: Two states are said to be physically equivalent if they can produce by the above method an orthonormal basis of the same Hilbert subspace, or in an equivalent way, if they belong to the same invariant subspace under the group action. It is easy to see that this is an equivalence relation. But if the representation is irreducible, all states are equivalent as basic pieces of physical information for describing the elementary system. There is one and only one single piece of basic physical information to describe an elementary object. That is what the term elementary means.

But this definition of elementary particle is a pure group theoretical one. The only

quantum mechanical ingredient of it is that the group operates on a Hilbert space. Then one question arises. Can we translate this QM definition into the classical domain and then obtain an equivalent group theoretical definition for a classical elementary particle? If this is the case we will be able to develop the $(s \neq 0)$ region of CM in a more equivalent way. The answer is affirmative.

In CM we have no vector space structure to describe the states of a system. What we have are manifolds of points where each point represents either the configuration state, the kinematical state or the phase state of the system depending on which manifold we work. But the idea that any point that represents the state of an elementary particle is physically equivalent to any other, is in fact the very mathematical concept of homogeneity of the manifold under the corresponding group action. In this way, the irreducibility assumption of the QM definition is translated into the realm of CM in the concept of homogeneity of the corresponding manifold under the Poincaré group or any other kinematical group we consider as the symmetry group of the theory. But, what manifold? Configuration space? Phase space? The answer as has been shown in previous papers [6], [7], [8], is that the appropriate manifold is the *kinematical space*.

In previous attempts, Bacry [9], considered the phase space as the homogeneous manifold to describe elementary objects arriving at the conclusion that the most general elementary particle is a system of four degrees of freedom. Three represent the position of the particle, being the linear momentum their conjugate variables, and the fourth is an angle α whose conjugate momentum is a spin component S_α such that the spin components are expressed in terms of these two variables and an invariant value S, the absolute value of the spin, in the form:

$$S_x = (S^2 - S_\alpha^2)^{1/2} \cos \alpha, \quad S_y = (S^2 - S_\alpha^2)^{1/2} \sin \alpha, \quad S_z = S_\alpha.$$

This result was generalized independently by Kirillov, Kostant and Souriau and is known as the KKS theorem, showing that the coadjoint action of any Lie group defines on its orbits a symplectic structure [10].

But the phase space, although interpreted as the state space of CM does not play the same role as the Hilbert space in QM at least as the dynamics is concerned. In QM the dynamics is stated in terms of initial $|\psi_i>$ and final $|\psi_f>$ states, such that the probability amplitude for the dynamical process $|\psi_i> \rightarrow |\psi_f>$ is given by the corresponding matrix element of the scattering operator $<\psi_f|S|\psi_i>$. But both $|\psi_i>$ and $|\psi_f>$ are elements of the same Hilbert space that at the same time it plays the role of the space that describes all the particle states, it also represents the kinematical space where the dynamics is running.

In the Lagrangian approach of CM the kinematical space is the manifold spanned by the initial (or final) boundary variables that are held fixed in the corresponding variational formulation. It is this manifold X where the dynamics is developed, where when quantizing the system we obtain the natural link between the classical and quantum formalisms through the Feynman's path integral approach, as shown in [8]. It is this manifold the natural base space to define on it the Hilbert space structure of the quantized system. In a formal way we can say that each point $x \in X$, of the kinematical space of the Lagrangian formalism that represents the inital or final kinematical state, smears out and is transformed through the Feynman's quantization into the squared integrable function $\psi(x)$ with support around that point x and representing the particle wave function.

Usually the Lagrangian of any classical system is restricted to depend only on the first order derivative of each of the variables q_i that represent the different independent degrees of freedom, or equivalently, that the q_i satisfy second order differential equations. But at this stage, if we do not know what are the basic variables we need to describe our elementary system how can we state that they necessarily satisfy second order differential equations? If some of the degrees of freedom, say q_1, q_2 and q_3, represent the center of mass position of the system, Newtonian mechanics implies that in this particular case L will depend on the first order derivatives of these three variables. But what about other degrees of freedom? It is the homogenenity condition on the kinematical space, as the mathematical statement of elementarity, that will restrict the dependence of the Lagrangian on these higher order derivatives. It is the definition of elementary particle that will suply the structure of the

Lagrangian.

A classical elementary particle is defined as a Lagrangian classical system whose kinematical space X, is a homogeneous space of the corresponding kinematical group G. [6–8]

Since the Galilei and Poincaré groups are 10-parameter Lie groups, the largest homogeneous space we can find is a 10-dimensional manifold whose variables share the same domain and dimensions like the variables we use to parameterize the group. Both groups are parameterized in terms of the following variables $(b, \mathbf{a}, \mathbf{v}, \alpha)$ with domains and dimensions respectively like $b \in R$ that represents the time parameter of the time translation, $\mathbf{a} \in R^3$ the three spacial coordinates for the space translation, $\mathbf{v} \in R^3$ the three components of the relative velocity of the inertial observers, that are restricted to $v < c$ in the Poincaré case, and finally $\alpha \in SO(3)$ are three angular variables that characterize the relative orientation of the corresponding Cartessian frames and whose compact domain is expressed in terms of a suitable parametrization of the rotation group.

In this way the maximum number of kinematical variables is also 10, and we represent them by $x \equiv (t, \mathbf{r}, \mathbf{u}, \alpha)$ with the same domains and dimensions as above and are interpreted respectively as the time, position, velocity and orientation of the particle. This is the same description of the initial and final states as the relativistic spherical top in the Hanson and Regge canonical approach [11], but in that work different constraints are used to avoid dependence of the Lagrangian on second order derivatives.

In our approach, since the Lagrangian must also depend on the next order derivatives of the kinematical variables we arrive at the conclusion that L must also depend on the acceleration and angular velocity of the particle. The particle is a system of six degrees of freedom, three, $\mathbf{r}$, represent the position of a point and other three α, its orientation in space, which we can visualize by assuming a system of three orthogonal unit vectors linked to point $\mathbf{r}$ as a body frame, but the Lagrangian may depend up to the second time derivative of $\mathbf{r}$, or acceleration of that point, and on the first derivative of α, i.e., the angular velocity. By this definition it is the kinematical group G that represents the special Relativity Principle that completely determines the structure of the Lagrangian that represents a clas-

sical elementary particle [6–8]. Point particles are particular cases of the above definition and their kinematical spaces are just the quotient structures between the group G and the homogeneous transformations subgroup of rotations and boosts, and thus their kinematical variables reduce only to time and position $(t, \mathbf{r})$.

We began this section by considering the classical limit of QM and we have remarked the different treatment the recent history of physics has done to the $(s \neq 0)$ region in both QM and CM formalisms. But this difference between the two approaches might have physical consequences. For instance, let us consider the well known QM diffraction experiment of a beam of particles by a certain slit. If we send a beam of point particles we expect in the screen a replica of the slit shape and size but if we obtain a diffraction pattern we interpret this outcome as a pure quantum mechanical effect. This interpretation is correct if for instance we perform the experiment with pions, i.e., spinless particles. But if the experiment is done with photons, electrons, neutrons, etc. as is usually the case, all of them are spinning particles. Since the spin of these particles is of order of $\hbar$ not the whole outcome of the experiment of order $\hbar$ is a quantum mechanical effect, because part of it must be related to the spin structure of the particles and must be analyzed theoretically. What is a quantum mechanical effect is the difference between the real outcome and that part that can be interpreted in classical terms but taking into account the corresponding classical spin contribution. A deeper understanding of the classical spin description will help to better understand the quantum mechanical effects.

In this work, and following with the above argument we shall consider the classical contribution to the tunnel effect. Classically, a point particle confined into a potential well can never escape from it if its energy is less than the top of the potential barrier. In QM there is a nonvanishing probability of crossing. However, if the particle has spin and this spin is of orbital nature as will be described in the next section, we shall see in Section V that because of the spin, the classical spinning particle can cross the barrier even with kinetic energies below the potential energy barrier, and that this contribution is not in general negligible when compared with the quantum tunnel effect.

III. CLASSICAL PARTICLE WITH (ANTI)-ORBITAL SPIN

We are going to analyze in detail a non-relativistic Lagrangian that although it does not give rise to a spin 1/2 particle when quantized (see [8]), however it behaves in most of the features simmilarly as the relativistic electron described in the mentioned reference, but the mathematics involved are simpler than in the general case, and illustrates the spin structure and dynamics for the example we want to work out in section V.

Let $\mathcal{G}$ be the Galilei group. Let us consider a Galilei particle whose kinematical space is $X = \mathcal{G}/SO(3)$, so that any point $x \in X$ can be characterized by the seven variables $x \equiv (t, \mathbf{r}, \mathbf{u})$, $\mathbf{u} = d\mathbf{r}/dt$, which are interpreted as the time, position and velocity of the particle respectively. In this example we have no orientation variables and because of this we cannot have, when quantizing the system, half integer spin values. The Lagrangian will also depend on the next order derivatives, i.e., on the acceleration of the particle. Rotation and translation invariance implies that L will be a function of only $\mathbf{u}^2$, $(d\mathbf{u}/dt)^2$ and $\mathbf{u} \cdot d\mathbf{u}/dt = d(u^2/2)/dt$, but being this last term a total derivative it will not be considered here.

Let us assume that our elementary system is represented by the following Lagrangian

$$L = \frac{m}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 - \frac{m}{2\omega^2} \left(\frac{d^2\mathbf{r}}{dt^2} \right)^2, \tag{1}$$

where m is the mass and the parameter ω of dimensions of s^{-1} represents an internal frequency. In terms of the kinematical variables and their derivatives, and in terms of some group invariant evolution parameter τ, the Lagrangian can be written as

$$L = \frac{m}{2} \frac{\dot{\mathbf{r}}^2}{\dot{t}} - \frac{m}{2\omega^2} \frac{\dot{\mathbf{u}}^2}{\dot{t}}, \tag{2}$$

where the dot means $\tau-$derivative. If we consider that the evolution parameter is dimensionless, all terms in the Lagrangian have dimensions of action. Because the Lagrangian is a homogeneous function of first degree in terms of the derivatives of the kinematical variables, L can also be written as

$$L = -T\dot{t} + \mathbf{R} \cdot \dot{\mathbf{r}} + \mathbf{U} \cdot \dot{\mathbf{u}}, \tag{3}$$

where the functions accompanying the derivatives of the kinematical variables are defined and explicitely given by

$$
\begin{aligned}
T &= -\frac{\partial L}{\partial \dot{t}} = \frac{m}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 - \frac{m}{2\omega^2} \left(\frac{d^2\mathbf{r}}{dt^2} \right)^2, \\
\mathbf{R} &= \frac{\partial L}{\partial \dot{\mathbf{r}}} = m \frac{d\mathbf{r}}{dt}, \\
\mathbf{U} &= \frac{\partial L}{\partial \dot{\mathbf{u}}} = -\frac{m}{\omega^2} \frac{d^2\mathbf{r}}{dt^2}.
\end{aligned}
\tag{4}
$$

Dynamical equations are:

$$\frac{1}{\omega^2} \frac{d^4\mathbf{r}}{dt^4} + \frac{d^2\mathbf{r}}{dt^2} = 0, \tag{5}$$

whose general solution is:

$$\mathbf{r}(t) = \mathbf{A} + \mathbf{B}t + \mathbf{C}\cos\omega t + \mathbf{D}\sin\omega t, \tag{6}$$

in terms of the 12 integration constants $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$.

When applying Noether's theorem to the invariance of dynamical equations under the Galilei group, the corresponding constants of the motion can be written in terms of the above functions in the form:

$$\text{Energy} \quad H = T - \mathbf{u} \cdot \frac{d\mathbf{U}}{dt} = \frac{m}{2}\mathbf{u}^2 - \frac{\omega^2}{2m}\mathbf{U}^2 - \mathbf{u} \cdot \frac{d\mathbf{U}}{dt}, \tag{7}$$

$$\text{Linear Momentum} \quad \mathbf{P} = \mathbf{R} - \frac{d\mathbf{U}}{dt} = m\mathbf{u} - \frac{d\mathbf{U}}{dt}, \tag{8}$$

$$\text{Galilei or Static Momentum} \quad \mathbf{K} = m\mathbf{r} - \mathbf{P}t - \mathbf{U}, \tag{9}$$

$$\text{Angular Momentum} \quad \mathbf{J} = \mathbf{r} \times \mathbf{P} + \mathbf{u} \times \mathbf{U} = \mathbf{L} + \mathbf{S}. \tag{10}$$

It is the presence of the $\mathbf{U}$ function that distinguishes the features of this system with respect to the point particle case. We see that the total linear momentum is not lying along the direction of the velocity $\mathbf{u}$. The translationally invariant part of the total angular momentum $\mathbf{J}$ is the spin of the system $\mathbf{S}$. It is the dependence on the acceleration that in this system of three degrees of freedom leads to the spin structure, such that the spin takes the form:

11

$$\mathbf{S} = \mathbf{u} \times \mathbf{U} = \frac{m}{\omega^2} \frac{d^2\mathbf{r}}{dt^2} \times \frac{d\mathbf{r}}{dt}, \tag{11}$$

is always orthogonal to the velocity and acceleration of point $\mathbf{r}$, and its dynamics is given by taking the time derivative of $\mathbf{J}$ in (10) and taking into account that $\mathbf{P}$ is constant it leads to

$$\frac{d\mathbf{S}}{dt} = \mathbf{P} \times \mathbf{u}. \tag{12}$$

The spin is not a constant of the motion except in the center of mass frame, it precesses along the linear momentum $\mathbf{P}$ such that its projection $\mathbf{S}\cdot\mathbf{P}$, called helicity, remains constant.

If we substitute the general solution (6) in (7-10) we see in fact that the integration constants are related to the above conserved quantities

$$H = \frac{m}{2}\mathbf{B}^2 - \frac{m\omega^2}{2}(\mathbf{C}^2 + \mathbf{D}^2),$$

$$\mathbf{P} = m\mathbf{B},$$

$$\mathbf{K} = m\mathbf{A},$$

$$\mathbf{J} = \mathbf{A} \times m\mathbf{B} - m\omega\mathbf{C} \times \mathbf{D}.$$

But at the same time that the particle acquires its spin we see that the static momentum $\mathbf{K}$ differs from the point particle case in the term $-\mathbf{U}$, such that if we define the vector $\mathbf{x} = \mathbf{U}/m$, then $\dot{\mathbf{K}} = 0$ leads from (9) to the equation:

$$\mathbf{P} = m\frac{d(\mathbf{r} - \mathbf{x})}{dt},$$

and $\mathbf{q} = \mathbf{r} - \mathbf{x}$, defines the position of the center of mass of the particle that is a different point than $\mathbf{r}$ and is given by

$$\mathbf{q} = \mathbf{r} + \frac{1}{\omega^2} \frac{d^2\mathbf{r}}{dt^2}, \tag{13}$$

such that in terms of it, dynamical equations can be separated into the form:

$$\frac{d^2\mathbf{q}}{dt^2} = 0, \tag{14}$$

$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2(\mathbf{r} - \mathbf{q}) = 0. \tag{15}$$

12

In terms of these variables the spin takes the form

$$\mathbf{S} = -(\mathbf{r} - \mathbf{q}) \times m\mathbf{u}, \tag{16}$$

and it has the formal expresion of the opposite of an orbital angular momentum with respect to point $\mathbf{q}$ of a point mass m at point $\mathbf{r}$ moving at the speed $\mathbf{u}$. We say that this spin is of (anti)-orbital nature.

Point $\mathbf{q}$ moves in a straight trajectory at constant velocity while the motion of point $\mathbf{r}$ is an isotropic harmonic oscillator of angular frequency ω around point $\mathbf{q}$. But if $\mathbf{q}$ represents the center of mass position then, what position does point $\mathbf{r}$ represent? Point $\mathbf{r}$ represents the charge of the particle position. This can be seen by considering some interaction with an external field. The homogeneity condition of the Lagrangian in terms of the derivatives of the kinematical variables leads us to consider an interaction term of the form

$$L_I = -e\phi(t, \mathbf{r})\dot{t} + e\mathbf{A}(t, \mathbf{r}) \cdot \dot{\mathbf{r}}, \tag{17}$$

linear in the derivatives and where the external potentials are functions of t and $\mathbf{r}$. We can also consider more general interaction terms depending on $\mathbf{u}$ and $\dot{\mathbf{u}}$, but this will not be an interaction with an external electromagnetic field. Then the dynamical equations are

$$\frac{1}{\omega^2}\frac{d^4\mathbf{r}}{dt^4} + \frac{d^2\mathbf{r}}{dt^2} = \frac{e}{m}\left(\mathbf{E}(t, \mathbf{r}) + \mathbf{u} \times \mathbf{B}(t, \mathbf{r})\right), \tag{18}$$

where the electric $\mathbf{E}$ and magnetic field $\mathbf{B}$ are expressed in terms of the potentials in the usual form, and because the interaction term does not affect the dependence of the Lagrangian on $\dot{\mathbf{u}}$, the spin and the center of mass definitions (11) and (13) respectively, remain the same as in the free particle case. Then dynamical equations can again be separated in the form

$$\frac{d^2\mathbf{q}}{dt^2} = \frac{e}{m}\left(\mathbf{E}(t, \mathbf{r}) + \mathbf{u} \times \mathbf{B}(t, \mathbf{r})\right), \tag{19}$$

$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2(\mathbf{r} - \mathbf{q}) = 0, \tag{20}$$

where the center of mass $\mathbf{q}$ satisfies Newton's equations under the action of the total external Lorentz force, while point $\mathbf{r}$ still satisfies the isotropic harmonic motion of angular frequency

ω around point $\mathbf{q}$. This internal motion remains unaffected by the external interaction. But the external force and the fields are defined at point $\mathbf{r}$ and not at point $\mathbf{q}$. Point $\mathbf{r}$ clearly represents the point charge position. In fact, this minimal coupling we have considered is the coupling of the electromagnetic potentials with the particle current, that in the relativistic case can be written as $A^\mu j_\mu$, but the current j_μ is associated to the motion of point $\mathbf{r}$. This charge has an oscillatory motion of very high frequency ω that in the case of the relativistic electron (see [7,8]), is $\omega = 2mc^2/\hbar \simeq 10^{20}\text{s}^{-1}$. The average position of the charge is the center of mass, but it is its internal rotational motion, usually known as the zitterbewegung, that gives rise to the spin structure and to the magnetic properties of the particle.

When analyzed in the center of mass observer (see Fig. 2), the system reduces to a point charge whose motion is in general an ellipse but if we choose $C = D$, and $\mathbf{C} \cdot \mathbf{D} = 0$, it reduces to a circle of radius $a = C = D$, orthogonal to the spin that is constant in this frame. Then if the particle has charge e, it has a constant magnetic moment

$$\mu = \frac{1}{2}\mathbf{r} \times \mathbf{j} = \frac{e}{2}\mathbf{r} \times \mathbf{u} = -\frac{e}{2m}\mathbf{S},$$

opposite to the spin direction if $e > 0$, and a nonvanishing oscilating electric dipole $\mathbf{p} = e\mathbf{r}$, orthogonal to both μ and $\mathbf{S}$ in this frame, such that its average value vanishes for times larger than the natural period of this internal motion.

In this case the above constants of the motion are no longer conserved quantities. For instance, the linear momentum satisfies

$$\frac{d\mathbf{P}}{dt} = \mathbf{F},$$

where $\mathbf{F}$ is the external Lorentz force. If we take the time derivative of the total angular momentum it leads to

$$\frac{d\mathbf{J}}{dt} = \mathbf{r} \times \frac{d\mathbf{P}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{P} + \frac{d\mathbf{S}}{dt}. \tag{21}$$

The spin dynamics is given by

$$\frac{d\mathbf{S}}{dt} = \frac{m}{\omega^2}\frac{d^3\mathbf{r}}{dt^3} \times \frac{d\mathbf{r}}{dt} = \mathbf{P} \times \mathbf{u}, \tag{22}$$

14

and remains the same expression as in the free case (12), but now $\mathbf{P}$ is not a constant function. If we substitute this in the previous expression (21), it turns out that the total angular momentum satisfies

$$\frac{d\mathbf{J}}{dt} = \mathbf{r} \times \mathbf{F}, \tag{23}$$

where on the right hand side we have the total external torque produced by the fields. Although the system has magnetic moment associated to the charge motion, because of the minimal coupling interaction, the external electromagnetic field produces forces, and the corresponding torques with respect to the origin, but no other torques, so that a spin evolution equation of the form

$$\frac{d\mathbf{S}}{dt} = \mu \times \mathbf{B},$$

as in Ref. [4], does not arise in this model in which the electromagnetic structure of the particle is just that of a point charge without any intrinsic magnetic moment, as it happens to be the case of the relativistic electron, where the charge motion in the center of mass frame is a circle of radius $a = S/mc = \hbar/2mc$ at the speed of light c. Only for anomalous magnetic moments we should have to consider the addition of external torques of the above form, where μ will be the anomalous magnetic moment. But in that case we need to have in the interaction Lagrangian terms of the form $F^{\mu\nu} M_{\mu\nu}/2$, where $M_{\mu\nu}$ is the dipole tensor that describes the intrinsic dipole structure of the particle.

If we consider that the mechanical energy of the particle is still given by expression (7), then because of the interaction it changes in the form

$$\frac{dH}{dt} = \mathbf{u} \cdot \mathbf{F} = e\mathbf{u} \cdot \mathbf{E}.$$

The change of the mechanical energy of the particle is the work of the external force along the charge trajectory. If the electric field is conservative then the change of this energy between two arbitrary points of the trajectory is

$$H(t_2) - H(t_1) = e\phi(t_1, \mathbf{r}_1) - e\phi(t_2, \mathbf{r}_2)$$

15

$$= e\left(\phi(t_1, \mathbf{q}_1) - \phi(t_2, \mathbf{q}_2)\right) + e\left(\phi(t_1, \mathbf{r}_1) - \phi(t_1, \mathbf{q}_1)\right) + e\left(\phi(t_2, \mathbf{q}_2) - \phi(t_2, \mathbf{r}_2)\right)$$

$$\simeq e\phi(t_1, \mathbf{q}_1) - e\phi(t_2, \mathbf{q}_2),$$

if the difference of potential between the center of mass and center of charge position at any instant can be considered negligible if the external field is of smooth variation.

Once we introduce the classical path into the free Lagrangian and integrate from the initial to the final point, we obtain the action function for the free system along this path that can be written in terms of the boundary kinematical variables in the form

$$\begin{aligned}
A(t_1, \mathbf{r}_1, \mathbf{u}_1; t_2, \mathbf{r}_2, \mathbf{u}_2) = \frac{m}{2\Delta}\bigg[&\frac{1}{\omega}(\mathbf{u}_2 - \mathbf{u}_1)^2 \sin\omega(t_2 - t_1) \\
&+ \omega(\mathbf{r}_2 - \mathbf{r}_1)^2 \sin\omega(t_2 - t_1) \\
&- (t_2 - t_1)(\mathbf{u}_2 - \mathbf{u}_1)^2 \cos\omega(t_2 - t_1) \\
&+ 2(t_2 - t_1)(\mathbf{u}_2 \cdot \mathbf{u}_1)(1 - \cos\omega(t_2 - t_1)) \\
&- 2(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{u}_1 + \mathbf{u}_2)(1 - \cos\omega(t_2 - t_1))\bigg],
\end{aligned}$$

where $\Delta = 2(1 - \cos\omega(t_2 - t_1)) - \omega(t_2 - t_1)\sin\omega(t_2 - t_1)$. This is the action function that must be considered when analyzing Feynman's kernel in the path integral approach of this system.

IV. CANONICAL FORMALISM

Although the Lagrangian depends on second order derivatives we can develop the corresponding canonical formalism. Starting from the Lagrangian

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 - \frac{m}{2\omega^2}\ddot{\mathbf{r}}^2,$$

where now the dot means time derivative, we have in this case six generalized coordinates that are not the independent degrees of freedom but they are the three degrees of freedom $\mathbf{q}_1 = \mathbf{r}$ and their first derivatives $\mathbf{q}_2 = d\mathbf{r}/dt$, such that the conjugate momenta are defined by [12]

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\mathbf{r}}}\right) = \mathbf{R} - \frac{d\mathbf{U}}{dt}, \quad \mathbf{p}_2 = \frac{\partial L}{\partial \ddot{\mathbf{r}}} = \mathbf{U}.$$

The phase space is a 12-dimensional manifold and the Hamiltonian is in fact the total energy written in terms of the canonical variables

$$H = \mathbf{p}_1 \cdot \dot{\mathbf{q}}_1 + \mathbf{p}_2 \cdot \dot{\mathbf{q}}_2 - L = \mathbf{p}_1 \cdot \mathbf{q}_2 - \frac{m}{2}\mathbf{q}_2^2 - \frac{\omega^2}{2m}\mathbf{p}_2^2.$$

Hamilton-Jacobi equations are

$$\dot{\mathbf{q}}_1 = \frac{\partial H}{\partial \mathbf{p}_1} = \mathbf{q}_2, \quad \dot{\mathbf{p}}_1 = -\frac{\partial H}{\partial \mathbf{q}_1} = 0,$$

$$\dot{\mathbf{q}}_2 = \frac{\partial H}{\partial \mathbf{p}_2} = -\frac{\omega^2}{m}\mathbf{p}_2, \quad \dot{\mathbf{p}}_2 = -\frac{\partial H}{\partial \mathbf{q}_2} = -\mathbf{p}_1 + m\mathbf{q}_2.$$

The ten Noether constants of the motion become in this formalism the generating functions of the corresponding canonical transformations of time and space translations, pure Galilei transformations and rotations and are explicitely given by

$$H = \mathbf{p}_1 \cdot \mathbf{q}_2 - \frac{m}{2}\mathbf{q}_2^2 - \frac{\omega^2}{2m}\mathbf{p}_2^2,$$

$$\mathbf{P} = \mathbf{p}_1,$$

$$\mathbf{K} = m\mathbf{q}_1 - \mathbf{p}_1 t - \mathbf{p}_2,$$

$$\mathbf{J} = \mathbf{q}_1 \times \mathbf{p}_1 + \mathbf{q}_2 \times \mathbf{p}_2,$$

and since the Poisson bracket of two constants of the motion is again a constant of the motion we obtain that the above constants of the motion satisfy the commutation relations

$$[J_i, J_k] = \epsilon_{ikl}J_l, \quad [J_i, P_k] = \epsilon_{ikl}P_l, \quad [J_i, K_k] = \epsilon_{ikl}K_l, \quad [J_i, H] = 0,$$

$$[P_i, P_k] = 0, \quad [P_i, H] = 0, \quad [K_i, K_k] = 0, \quad [K_i, H] = P_i, \quad [K_i, P_j] = m\delta_{ij},$$

where $[.,.]$ is the corresponding Poisson bracket and because $[K_i, P_j] \neq 0$ they are not the commutations relations of the Galilei group but rather those of the extended Galilei group

[13]. Although K_i satisfies $[K_i, H] = P_i$, it is a constant of the motion because as we see in (9) that $\mathbf{K}$ is time dependent its total time derivative is

$$\frac{d\mathbf{K}}{dt} = [\mathbf{K}, H] + \frac{\partial \mathbf{K}}{\partial t} = \mathbf{p}_1 - \mathbf{p}_1 = 0.$$

The spin observable $\mathbf{S} = \mathbf{u} \times \mathbf{U} = \mathbf{q}_2 \times \mathbf{p}_2$ satisfies the commutation relations

$$[S_i, S_k] = \epsilon_{ikl} S_l, \quad [J_i, S_k] = \epsilon_{ikl} S_l, \quad [S_i, P_k] = 0,$$

$$[S_i, K_j] = -\epsilon_{ijk} U_k = -\epsilon_{ijk} p_{2k}, \quad [S_i, H] = \frac{dS_i}{dt} = (\mathbf{P} \times \mathbf{u})_i,$$

showing respectively that it is an angular momentum, that transforms like a vector under rotations, it is invariant under space translations but not under pure Galilei transformations and is not a constant of the motion, but satisfies the dynamical equations (12). We can check that the two constants of the motion

$$E = H - \frac{\mathbf{P}^2}{2m}, \quad \text{and} \quad Z^2 = \left(\mathbf{J} - \frac{1}{m}\mathbf{K} \times \mathbf{P}\right)^2, \tag{24}$$

commute with the above ten generators and they are Galilei invariant properties of the particle, that together with the mass m completely characterize the structure of this particle. They are the three functionally independent Casimir invariants of the extended Galilei group. In fact E is the Galilei internal energy of the particle and Z^2 is the squared of the total angular momentum or spin for the center of mass observer [13].

V. CLASSICAL TUNNEL EFFECT

Let us consider a spinning particle as described above under the influence of a potential barrier. Sharp walls correspond classically to infinite forces so that we shall consider potentials that give rise to finite forces like those of the shape depicted in Fig. 3, where V_0 represents the top of the potential. Then the external force $F(x)$, is constant and directed leftwards in the region $x \in (-a, 0)$ and directed rightwards for $x \in (0, b)$, vanishing outside these regions.

Let us assume for simplicity that the spin is pointing up or down in the z direction such that the point charge motion takes place in the XOY plane. Let q_x, q_y and $q_z = 0$, be the coordinates of the center of mass and x, y and $z = 0$, the position of the charge.

Dynamical equations are

$$\frac{d^2 q_x}{dt^2} = \frac{1}{m} F(x), \quad \frac{d^2 q_y}{dt^2} = 0, \tag{25}$$

$$\frac{d^2 x}{dt^2} + \omega^2 (x - q_x) = 0, \quad \frac{d^2 y}{dt^2} + \omega^2 (y - q_y) = 0, \tag{26}$$

where

$$F(x) = \begin{cases} -eV_0/a, & \text{for } x \in (-a, 0), \\ eV_0/b, & \text{for } x \in (0, b), \\ 0, & \text{otherwise.} \end{cases}$$

To make the corresponding numerical analysis we shall define different dimensionless variables. Let R be the average separation between the center of charge and center of mass. Then we define the new dimensionless position variables:

$$\hat{q}_x = q_x/R, \quad \hat{q}_y = q_y/R, \quad \hat{x} = x/R, \quad \hat{y} = y/R, \quad \hat{a} = a/R, \quad \hat{b} = b/R.$$

The new dimensionless time variable $\alpha = \omega t$ is just the phase of the internal motion, such that dynamical equations become

$$\frac{d^2 \hat{q}_x}{d\alpha^2} = A(\hat{x}), \quad \frac{d^2 \hat{q}_y}{d\alpha^2} = 0,$$

$$\frac{d^2 \hat{x}}{d\alpha^2} + \hat{x} - \hat{q}_x = 0, \quad \frac{d^2 \hat{y}}{d\alpha^2} + \hat{y} - \hat{q}_y = 0,$$

where $A(x)$ is given by

$$A(\hat{x}) = \begin{cases} -eV_0/\hat{a}m\omega^2 R^2, & \text{for } \hat{x} \in (-\hat{a}, 0), \\ eV_0/\hat{b}m\omega^2 R^2, & \text{for } \hat{x} \in (0, \hat{b}), \\ 0, & \text{otherwise.} \end{cases}$$

19

In the case of the electron, the internal velocity of the charge is $\omega R = c$, (see [7] and [8]), so that the parameter $e/mc^2 = 1.9569 \times 10^{-6} \mathrm{V}^{-1}$, such that for potentials of order of 100 volts we can take the dimensionless parameter $eV_0/m\omega^2 R^2 \simeq 2 \times 10^{-4}$.

If we choose as initial conditions for the center of mass motion

$$\hat{q}_y(0) = 0, \quad d\hat{q}_y(0)/d\alpha = 0,$$

then the center of mass is moving along OX axis. Then the above system reduces to the analysis of the one-dimensional motion where the only variables are $\hat{q}_x$ and $\hat{x}$. Let us call from now on these variables q and x respectively and remove all hats from the dimensionless variables. Then the dynamical equations to be solved numerically are just

$$\frac{d^2 q}{d\alpha^2} = A(x), \quad \frac{d^2 x}{d\alpha^2} + x - q = 0, \tag{27}$$

where $A(x)$ is given by

$$A(\hat{x}) = \begin{cases} -2a^{-1}10^{-4}, & \text{for } x \in (-a, 0), \\ 2b^{-1}10^{-4}, & \text{for } x \in (0, b), \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

Numerical integration has been performed by means of the computer package *ODE Workbench* [14]. The quality of the numerical results is tested by using the different integration schemes this program allows, ranging from the very stable embedded Runge-Kutta code of eight order due to Dormand and Prince to very fast extrapolation routines. All codes have adaptive step size control and we check that smaller tolerances do not change the results.

With $a = b = 1$, and in energy units such that the top of the barrier is 1, if we take an initial kinetic energy K below this threshold, $K = m\dot{q}(0)^2/2eV_0 = 0.41$ we obtain for the center of mass motion the graphic depicted in Fig. 4 where it is shown the variation of the kinetic energy of the particle $K(q)$, with the center of mass position during the crossing of the barrier. There is always crossing with a kinetic energy above this value. In Fig. 5 the same graphical evolution with $a = 1$ and $b = 10$ and $K = 0.8055$ for a potential of 10^3 Volts in which the different stages in the evolution are more evident. Below the initial values

20

for the kinetic energy of 0.4 and 0.80 respectively, the particle does not cross the potential barrier and it is rejected backwards.

If in both examples the parameter a is ranged from 1 to 0.05, thus making the left slope sharper, there is no appreciable change in the crossing energy, so that with $a = 1$ held fixed we can compute the minimum crossing energies for different b values, $K_c(b)$. Initial center of mass position has been ranged from $q(0) = -3.5$ to -3.0, while the center of charge initial position and velocity from $x(0) = -2.5$ to -2.0, and $\dot{x}(0) = \dot{q}(0)$, respectively.

To compare this model with the quantum tunnel effect, let us consider a point particle of mass m and charge e, under the same one-dimensional potential depicted in Fig. 3. If the particle of total energy E, is initially on the left hand side of the barrier, the wave function of this system is:

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & \text{for } x \leq -a, \\ C_1\text{Ai}(D(1 - G + x/a)) + C_2\text{Bi}(D(1 - G + x/a)), & \text{for } -a \leq x \leq 0, \\ C_3\text{Ai}(L(1 - G - x/b)) + C_4\text{Bi}(L(1 - G - x/b)), & \text{for } 0 \leq x \leq b, \\ Te^{ikx}, & \text{for } x \geq b, \end{cases} \tag{29}$$

where x is the same dimensionless position variable as before, and the constants

$$k = \left(\frac{E}{2mc^2}\right)^{1/2}, \quad D = \left(\frac{eV_0a^2}{2mc^2}\right)^{1/3}, \quad L = \left(\frac{eV_0b^2}{2mc^2}\right)^{1/3}, \quad G = \frac{E}{eV_0}, \tag{30}$$

and $\text{Ai}(x)$ and $\text{Bi}(x)$ are the Airy functions of x. The six integration constants R, T, and $C_i, i = 1, 2, 3, 4$, can be obtained by assuming continuity of the functions and their first order derivatives at the separation points of the different regions. The coefficient $|R|^2$ represents the probability of the particle to be reflected by the potential and $|T|^2$ its probability of crossing.

Computed the T variable for $a = 1$ and different values of the potential width b, and for energies below the top of the barrier eV_0 we show on Fig. 6 the average probability for quantum tunneling for four different extraction potentials V_0 of 10^2, 10^3, 10^4 and 10^5 Volts. This average probability has been computed by assuming that on the left of the barrier there is a uniform distribution of particles of energies below eV_0.

Let us consider for the classical spinning particle the same uniform distribution of particles. Then, the function $P(b) = 1 - K_c(b)$, where $K_c(b)$ is the minimum kinetic energy for crossing computed before, represents the ratio of the electrons that with kinetic energy below the top of the potential, cross the barrier because of the spin contribution. This function is also depicted in Fig. 6, and clearly has a decreasing behaviour with the distance b. The different barriers we have considered so far, can be interpreted as the one-dimensional extraction potentials of a probe tip placed at a variable distance b and at a constant difference of potential V_0 with respect to the sample. We see that for the different potentials shown in that figure this probability of crossing is greater in the quantum case than in the classical spin contribution, but they are geting closer for higher potentials.

The main feature of this model that contributes to tunneling is that while the center of mass is still on the left hand part of the barrier and since the cyclic internal motion of the charge is unchanged by the interaction this implies that once the charge penetrates into the left slope of the potential during a fraction of its cycle the kinetic energy is decreased, while during the remaining time of its cycle it is unaffected by external forces so that the center of mass motion is a kind of short decceleration periods under a constant force with short periods of constant velocity in between as is clearly visible in the saw-teeth shape of the particle kinetic energy of Fig. 5. It turns out that the average deccelerating force acting on the center of mass of the particle is smaller than in the case of a pure point particle. When the particle has completely penetrated into the potential and not all its kinetic energy has been exhausted, then if the penetration is sufficient such that the charge position can reach during part of its cycle the right slope of the potential, it turns out that during that time the center of mass is under a force directed to the right and thus the kinetic energy is increased. It is clear that at least one of the regions $(-a, 0)$ or $(0, b)$ must be of the order of the radius of the internal motion to have an appreciable crossing. In the case of the electron model (see [7] and [8]) this radius is $\hbar/2mc$, i.e., half the Compton wave-length of the electron, so that with a or b ot this order of magnitude as has been considered in the above examples, we can have a nonvanishing crossing.

22

We see that the separation between the center of mass and center of charge that gives rise to the spin structure of this particle model justifies that this system can cross a potential barrier even if its kinetic energy is below the top of the potential.
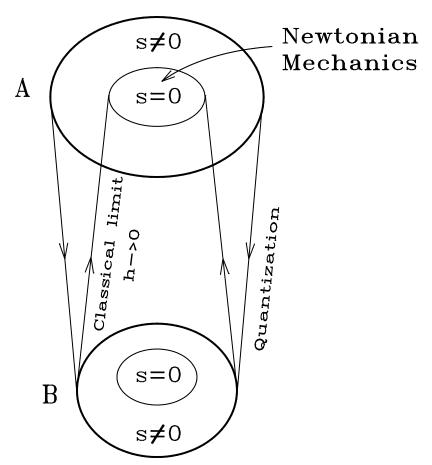
ACKNOWLEDGMENTS

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A.Messiah, *Mécanique Quantique*, (Dunod, Paris, 1959) ch. VI p. 180:

"à la limite où $\hbar \rightarrow 0$, les lois de la Mécanique Quantique doivent se réduire à celles de la Mécanique Classique".

E. Merzbacher, *Quantum Mechanics*, (John Wiley, N.Y., 1970), p. 3:

"We thus expect that classical mechanics is contained in quantum mechanics as a limiting form $(\hbar \rightarrow 0)$".

L. Landau and E. M. Lifshitz, *Quantum Mechanics*, (Pergamon, Oxford, 1974), p. 25:

"The transition from quantum mechanics to classical mechanics, corresponding to large phase, can be formally described as a passage to the limit $\hbar \rightarrow 0$...".

J. Glimm and A. Jaffe, *Quantum Physics*, (Springer Verlag, Berlin, 1987), p. 3:

"Classical mechanics is the limit $\hbar \rightarrow 0$ of quantum mechanics. Nonrelativistic (Newtonian) mechanics is the limit $c \rightarrow \infty$ of special Relativity."

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FIGURES

FIG. 1. Classical limit of Quantum Mechanics.

FIG. 2. Charge motion in the Center of Mass frame.

FIG. 3. Potential barrier to be crossed by spinning particles.

FIG. 4. Evolution of the kinetic energy during the crossing of the barrier with $a = b = 1$, initial energy $K = 0.41$ and extraction potential $V_0 = 100$ Volts.

FIG. 5. Evolution of the kinetic energy during the crossing of the barrier with $a = 1$, $b = 10$, initial energy $K = 0.91$ and extraction potential $V_0 = 1000$ Volts.

FIG. 6. Classical probability of tunneling $P(b)$ and quantum tunneling for four different extraction potentials V_0.

Classical Mechanics

s≠0

Newtonian
Mechanics

s=0

A

Classical limit
h—>0

Quantization

B

s=0

s≠0

Quantum Mechanics

u

e

eVeV

$eV(x)$

μ

r

a

C.M.

S

x

$-a$ O b

eV_O

$eV(x)$

$exp(-0.92\ b^{0.38})$

$K_c(b)^{-a}$

O

b

x

b